Yiddish word of the day

"der visnshaftler" = דער וויסנשאַפֿטלער

a scientist =

Yiddish phrase

"es iz ___ in droysn" = עס איז ___ אין דרויסן

the weather is ___

Rank - Nullity Thrm

Recall: A $m \times n$ matrix

- $\text{null}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n : A\vec{x} = \vec{0} \}$

  = solution set to the homogeneous system

- We saw that this was a subspace
  - so in particular null(A) has basis

Def: The dimension of null(A) is called nullity of A

$$\text{nullity}(A) = \dim(\text{null}(A))$$

Recall: the # free variables is the # vectors in the basis for null(A)

$\Rightarrow$ nullity(A) = # columns without a pivot.

- col(A) = span(columns of matrix A)

  = the vectors $\vec{b}$ in $\mathbb{R}^m$ such the matrix equation $A\vec{x} = \vec{b}$ has a solution.

Def: The rank of matrix A is

rank(A) = dim(col(A))

- Since col(A) = span($v_1 \downarrow$ . . . $v_n \downarrow$)

The # of linearly independent in this list $v_1 \dots v_n$ will be the # of vectors in the basis.

• This implies $\text{rank}(A)$ = # of leading variables
= # of columns with a pivot!!

Note: A $m \times n$ matrix
• n = # of columns

$\Rightarrow n$ = # columns with pivot + # columns of columns w/out a pivot

$$\boxed{n = \text{rank}(A) + \text{nullity}(A)}$$

Rank-Nullity theorem for matrices.

ex) Let A be a 6×5 matrix and the null space of A is

$$\text{null}(A) = \text{span}\left(\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}, \begin{pmatrix}9\\0\\1\\0\\0\end{pmatrix}\right)$$

Q: What is rank of A?

• nullity $A = 2$ $\Rightarrow$ $5 = \text{rank}(A) + 2$ $\Rightarrow$ $\text{rank}(A) = 3$

ex) A 3×8 matrix with $\text{nullity}(A) = 6$.

Do the columns of A span $\mathbb{R}^3$?

No! $8 = \text{rank}(A) + 6$ $\Rightarrow$ $\text{rank}(A) = 2$.

Need at least 3 to span.

Chapter 6 - Linear Transformations

Def: V, W be vector spaces. Then a function

$T: V \to W$ is a linear transformation if

1) $T(u+v) = T(u) + T(v)$ for u, v in V

2) $T(cu) = cT(u)$ for c in $\mathbb{R}$, u in V

ex) $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d = (\text{trace of matrix})$$

$$\cdot\, T\left(\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{M_1} + \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}_{M_2}\right) = T\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} = a_{11}+b_{11}+a_{22}+b_{22} = T(M_1)+T(M_2)$$

ii) $D : \mathbb{R}_n[x] \longrightarrow \mathbb{R}_{n-1}[x]$ Check this is a LT

$$D(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

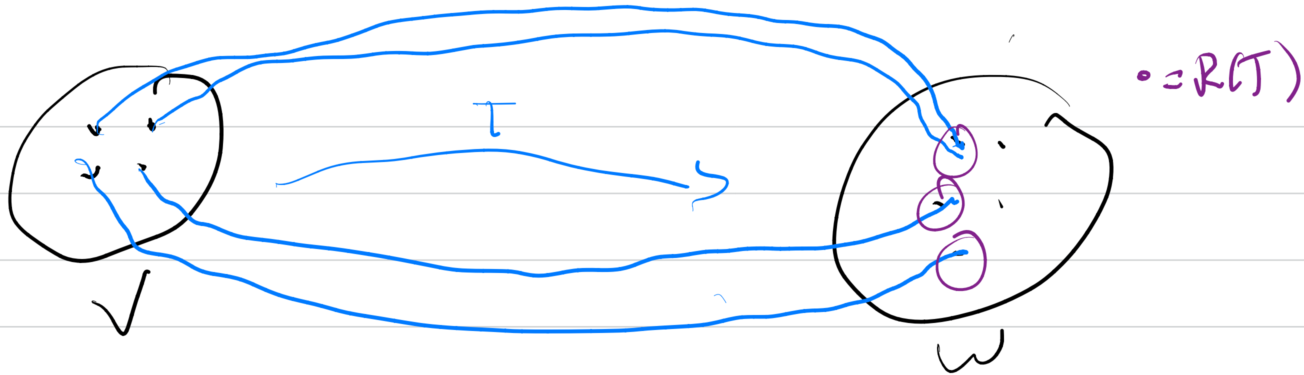
iii) $D(f) : C^\infty(\mathbb{R}^n; \mathbb{R}^m) \longrightarrow M_{m \times n}(\mathbb{R})$

$$f : \mathbb{R}^n \to \mathbb{R}^m \longrightarrow Df \text{ is a } M \times n \text{ matrix}$$

Def: $T : V \to W$ be linear transformation.

1) $Ker(T) = \{ v \text{ in } V : T(v) = 0_w \}$ (Kernal of T)

2) Range of T, $R(T) = \{ w \text{ in } W :$ there is a v in V with $T(v) = w \}$



Recall: If $T: \mathbb{R}^n \to \mathbb{R}^m$ linear we saw that T was uniquely defined on the Standard Basis

· Same is true for $T: V \to W$

In particular: Let $\mathcal{B}_V = (v_1, \dots, v_n)$ be a basis for V

then $R(T) = \text{span}\left(T(v_1), T(v_2), \dots, T(v_n)\right)$

- So in particular, a basis for the range is just those vectors in $(T(v_1), \ldots, T(v_n))$ that are LI.

ex) $T: \mathbb{R}_2[x] \to M_{2\times 2}(\mathbb{R})$ $\quad \mathcal{B}_V = (1, x, x^2)$

$$T(a+bx+cx^2) = \begin{pmatrix} a & a+b+c \\ c & -b \end{pmatrix} \qquad \mathcal{B}_{M_{2\times 2}(\mathbb{R})} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & -b \end{pmatrix} + \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= aT(1) + bT(x) + cT(x^2)$$

So to find basis for $R(T)$, check which of the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are LI

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So basis for range is $\mathcal{B} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$

More on Kernel

ex) $D : \mathbb{R}_3[x] \to \mathbb{R}_2[x]$ the derivative.

- What is $\ker(D)$ = all constant polynomials

= span(1)

ex) $T(a+bx+cx^2) = \begin{pmatrix} a & a+b+c \\ c & -b \end{pmatrix}$

$\ker(T) = \left\{ a+bx+cx^2 : \begin{pmatrix} a & a+b+c \\ c & -b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \{ 0 \}$

Prop: T linear $T(0_V) = 0_W$ (show!!)

Def: $T: V \to W$ linear transf

1) nullity of T is $\dim \ker(T)$

2) rank of T is $\dim R(T)$

Recall: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear $\longrightarrow$ A_T $m \times n$ matrix

such that $T(\vec{x}) = A_T \vec{x}$

$T: \mathbb{R}^n \to \mathbb{R}^m$	A_T $m \times n$ matrix
Kernel of T $\ker(T)$	$\text{null}(A_T)$
$\text{nullity}(T)$	$\text{nullity}(A_T)$
range of T $R(T)$ =	$\text{Col}(A_T)$
$\text{rank}(T)$ =	$\text{rank}(A_T)$

In the case of $T: \mathbb{R}^n \to \mathbb{R}^m$ we saw that

$\text{rank}(T) + \text{nullity}(T) = n$

Is this true in general?

Let V n-dim VS, W m-dimensional VS.

$T: V \to W$ linear transf.

Then $n = \dim(V) = \text{rank}(T) + \text{nullity}(T)$

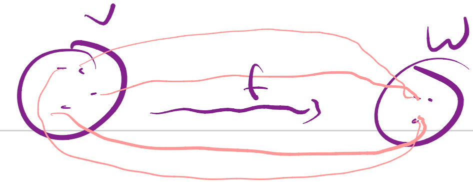
$\boxed{\text{Rank-Nullity Thrm for Linear transformations!}}$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & H \\ G/\ker f \arrow[ur, dashed] & \end{tikzcd}$$

1st Isomorphism Thrm

• the cover of our canvas page!!

Def: $f: V \to W$ any function



1) We say f is surjective (onto) if $R(f) = W$

2) We say f is injective (1-1) if whenever $f(x_1) = f(x_2)$ then $x_1 = x_2$

Thrm: $T: V \to W$ be linear transf.

Then T is injective if and only if $\boxed{\ker T = \{0\}}$

ex) $T(a+bx+cx^2) = \begin{pmatrix} a & a+b+c \\ c & -b \end{pmatrix}$

- We saw $\ker(T) = \{0\}$ so T is injective

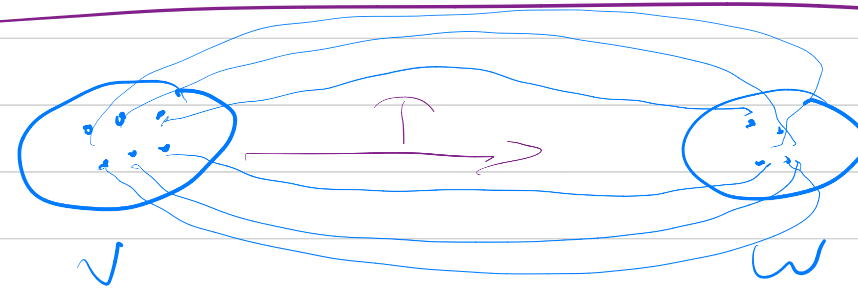
  but T is not surjective

  ($R(T)$ has $\dim = 3$, but $M_{2\times 2}(\mathbb{R})$ has $\dim = 4$)

We will adress how to check for surjective/injective in a second, but first we have some applications of rank-nullity.

Thm: V n-dim, W m-dim $T: V \to W$ linear

1) If $\dim V > \dim W$ then T is NOT injective



$\dim V = 6$ $\qquad$ $\dim W = 4$

$$\begin{aligned} n &= \operatorname{rank}(T) + \operatorname{nullity}(T) \\ &\le m + \operatorname{nullity}(T) \\ 0 < n - m &\le \operatorname{nullity}(T) \end{aligned} \quad \Rightarrow \operatorname{nullity}(T) > 0$$

2) If $\dim V < \dim W$ then T is NOT surjective.

Pf Uses Rank-Nullity

STOP HERE!

3) <u>Def</u>: We say T is an isomorphism if T is both injective and surjective.

- If $T: V \to W$ is an isomorphism we say V is isomorphic to W (write $V \cong W$)

Note if T is injective, need $\dim V \leq \dim W$
if T is surjective, need $\dim V \geq \dim W$

So if T is an isomorphism we have $\dim V = \dim W$

Thrm: Two vectors are isomorphic if and only if

ex) $M_{m \times n}(\mathbb{R}) \cong \qquad \cong$

$M_{3 \times 2}(\mathbb{R}) \cong \qquad \cong$